CECS-PHY-05-10
hep-th/0512074

Exploring AdS Waves Via Nonminimal Coupling*

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We consider nonminimally coupled scalar fields to explore the Siklos spacetimes in three dimensions. Their interpretation as exact gravitational waves propagating on AdS restrict the source to behave as a pure radiation field. We show that the related *pure radiation constraints* single out a unique self-interaction potential depending on one coupling constant. For a vanishing coupling constant, this potential reduces to a mass term with a mass fixed in terms of the nonminimal coupling parameter. This mass dependence allows the existence of several free cases including massless and tachyonic sources. There even exists a particular value of the nonminimal coupling parameter for which the corresponding mass exactly compensates the contribution generated by the negative scalar curvature, producing a genuinely massless field in this curved background. The self-interacting case is studied in detail for the conformal coupling. The resulting gravitational wave is formed by the superposition of the free and the self-interaction contributions, except for a critical value of the coupling constant where a non-perturbative effect relating the strong and weak regimes of the source appears. We establish a correspondence between the scalar source supporting an AdS wave and a *pp* wave by showing that their respective pure radiation constraints are conformally related, while their involved backgrounds are not. Finally, we consider the AdS waves for topologically massive gravity and its limit to conformal gravity.

PACS numbers: 04.60.Kz, 04.50.+h, 04.30.Db

I. INTRODUCTION

This year was declared by the UNESCO the international year of Physics due to the seminal contributions made a century ago by Einstein to the modern understanding of nature. The legacy of Einstein not only resides on his successful ideas, he has also been influential through of its apparent mistakes. His "biggest blunder," as he called it, was the introduction of a cosmological constant. However, our view on this subject is different today where the applications of the ideas related with the cosmological term rank from quantum field theory to cosmology, and from black holes to holographic proposals of quantum gravity, just for citing a few examples.

In this work we shall concentrate in other no-less-important application: the propagation

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of gravitational waves in presence of a cosmological constant. The pioneering studies on this subject started with the work of García and Plebański [1] followed by references [2, 3, 4]. They generalized the Kundt and Robinson-Trautman vacuum gravitational waves [5, 6] to the case where the cosmological constant is different from zero (for a review, see Ref. [7]). The spacetimes introduced in Refs. [1, 2, 3, 4] are algebraically special and contain an interesting symmetrical class where the multiple principal null direction k^μ, corresponding to their Weyl tensor, is additionally a Killing vector. This class is diffeomorphic to the so-called Siklos spacetimes [8], which require a negative cosmological constant.

In D-dimensions the Siklos spacetimes can be defined by the following conformal transformation of a *pp* wave background

$$ds^2 = \frac{l^2}{y^2}\left[-F(u,y,x^i)du^2 - 2dudv + dy^2 + dx_i dx^i\right], \tag{1}$$

where i ranges from 1 to $D-3$. Here the null Killing field is given by $k^\mu\partial_\mu = \partial_v$. Note that in four dimensions, the only Einstein spaces conformally related to a *pp* wave geometry with a smooth function F are characterized by the above metric [8].[1] For a vanishing structural function $F = 0$, we recover the anti-de Sitter space metric, while for $F \ll 1$ this metric describes just a perturbation of AdS. In fact, the metric (1) can also be obtained from the AdS one by a generalized Kerr-Schild transformation

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{AdS}} - \frac{y^2 F}{l^2} k_\mu k_\nu, \tag{2}$$

in an analogous way than *pp* wave backgrounds are obtained from flat metric by a standard Kerr-Schild transformation. Hence, the AdS waves are to AdS space what *pp* waves are to Minkowski space. For a more precise interpretation of the Siklos spacetimes as exact gravitational waves propagating on AdS space, see Ref. [10].

The wave fronts of these AdS waves, defined by the $(D-2)$-surfaces $u, v = \text{const.}$, are given by hyperboloids with constant curvature proportional to $-1/l^2$. Additionally, the null Killing field ∂_v is geodesic but is not covariantly constant or parallel. Consequently, the AdS waves are neither *plane* fronted nor have *parallel* rays as their cousin configurations the *pp* waves. For this reason we find misleading the term "AdS *pp* waves" commonly used in the recent literature to characterize the gravitational fields (1), and instead we will refer to them as "AdS waves" throughout this paper. This term is also inaccurate since in general the Siklos spacetimes are not the only exact gravitational waves propagating on the AdS background, but we find less harmful to use vague terminology in comparison to use an incorrect one.

In this paper we are interested in lower dimensional configurations. For $D = 3$, which is the case of our interest, the metric form of the AdS waves (1) is preserved under the following coordinate transformations

$$(u, v, y) \mapsto \left(\tilde{u} = \int \frac{\mathrm{d}u}{f^2}, \tilde{v} = v - \frac{1}{2f}\frac{\mathrm{d}f}{\mathrm{d}u}y^2 + \frac{1}{2}\int \mathrm{d}u F_0, \tilde{y} = \frac{y}{f}\right), \tag{3}$$

together with the redefinition of the structural function

$$\tilde{F} = f^2(F - F_2 y^2 - F_0). \tag{4}$$

[1] "Impulsive" *pp* waves, i.e. those allowing a distributional dependence on retarded time, can be also conformally related to Einstein spaces with positive curvature [9].

Here $f = f(u)$ and $F_0 = F_0(u)$ are two arbitrary functions of the retarded time, and the coefficient $F_2 = F_2(u)$ is defined by the equation

$$F_2 = -\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}. \tag{5}$$

The above transformations are the 3-dimensional version of those found by Siklos in 4-dimensions [8]. They are behind of the conformal asymptotic symmetry observed in asymptotically AdS_3 spacetimes [11], and can be generalized to D-dimensions and associated to a central extension of the Virasoro symmetry [12].

It follows from the previous transformations that, for a general function F, any quadratic and zeroth order dependence on the wave-front coordinate y can be locally eliminated by a coordinate transformation. Indeed, for a given quadratic coefficient F_2 we just need to insert it in the left hand side of Eq. (5) and to impose that the function f in the coordinate transformation (3) satisfies the resulting differential equation. We would like to emphasize that the two spacetimes related by the above procedure are only equivalent at the local level, since their corresponding global behavior could be drastically different; in general the new coordinates in transformation (3) run over ranges which are different from the ranges of the starting coordinates. This remark is of exceptional transcendence in $2+1$ dimensions and is one of the lessons exhibited by the discovery of the BTZ black hole [13], i.e. the physically meaning vacuum configuration in $2+1$ gravity is just a proper identification of AdS_3 [14]. However, in this work we just concentrate on local issues and the consequences of the diverse global behaviors of the AdS waves presented here, in spite of being an interesting topic by itself, is beyond the scope of the present paper.

In vacuum $2+1$ AdS gravity, the resulting AdS waves are trivial in the sense that they can be cast as AdS_3 spaces using the above coordinate transformations. This in turn imply that a matter source must be introduced in order to support these configurations. In this work, we show that a nonminimally coupled scalar field presents many interesting features to be considered as a source. Moreover, this work is a natural extension of previous ones where the same kind of matter is nonminimally coupled to *pp* waves but where the presence of a cosmological constant is generically forbidden [15, 16, 17, 18].

The paper is organized as follows. In the next section, we derive the independent field equations governing the generation of AdS waves by nonminimally coupled scalar fields. This self-gravitating process is possible only if the scalar field is constrained by the fact that all its energy-momentum components vanish except the energy density along the retarded time, i.e. the scalar field must behave like a pure radiation field. In Sec. III, we show that the resulting *pure radiations constraints* uniquely determine the scalar configuration, not only by fixing the dependence of the field but also by selecting a unique self-interaction potential for any value of the nonminimal coupling parameter ξ. This potential depends on a single coupling constant λ, and for $\lambda = 0$ it becomes a mass term. The related free configurations, studied in Sec. IV, are characterized by the unusual feature that their mass is fixed by the nonminimal coupling parameter. This section is divided in two parts where in the first subsection IV A we determine the AdS waves corresponding to generic values of the nonminimal coupling parameter. The second subsection IV B is deserved to the study of the massless cases, which correspond to the minimal coupling ($\xi = 0$), and the conformal couplings in three ($\xi = 1/8$) and four ($\xi = 1/6$) dimensions. Due to the nonminimal coupling to gravity only the minimal case $\xi = 0$ describes a genuinely massless field on these curved backgrounds while for $\xi = 1/8$ and $\xi = 1/6$ the mass acquires a tachyonic contribution due

to the negative scalar curvature. For the specific value $\xi = 1/5$, this tachyonic contribution is exactly canceled out and consequently the scalar field becomes a genuinely massless field on a curved background. In Sec. V, we analyze the case where the scalar field is self-interacting ($\lambda \neq 0$) taking as explicit example the conformal coupling $\xi = 1/8$. In this section we derive the related AdS wave configuration and show that this later consistently reduces to the free configuration as the coupling constant λ is put to zero. In fact, the background corresponds to a double Kerr-Schild transformation of AdS, where the first transformation is just the free field contribution and the other one depends on the self-interaction. Additionally, we found a critical value of the coupling constant given by $\lambda = (\kappa/8l)^2$, for which a non-perturbative effect relating the strong and weak regimes of the sources appears. Finally, in Sec. VI we establish a correspondence between the AdS wave scalar field configurations and those supporting a *pp* wave. It is shown that, starting from a *pp* wave scalar source, i.e. the scalar field and the potential which satisfy the pure radiation constraints on the *pp* wave background, one can generate the AdS wave source and *vice et versa*. The appendix is devoted to the free case, when the AdS waves are ruled by topologically massive gravity with a negative cosmological constant. Conformal gravity configurations are also obtained as a zero topological mass limit of these configurations.

II. NONMINIMALLY COUPLED SCALAR FIELD SUPPORTING AN ADS WAVE

In this work we are concerned with scalar fields nonminimally coupled to an AdS wave background. The field equations are those arising from the variation of the following action

$$S = \int d^3x\,\sqrt{-g}\left(\frac{1}{2\kappa}(R+2l^{-2}) - \frac{1}{2}\nabla_\alpha\Phi\nabla^\alpha\Phi - \frac{1}{2}\xi R\,\Phi^2 - U(\Phi)\right), \tag{6}$$

where $\Lambda = -l^{-2}$ is the negative cosmological constant and ξ is the parameter allowing a nonminimal coupling to gravity of the scalar field Φ. The potential $U(\Phi)$ denotes a possible self-interaction whose form, as we shall see later, is dictated by the field equations. The variation of the above action with respect to the metric and the scalar field yield to the Einstein and the nonlinear Klein-Gordon equations, respectively,

$$G_{\alpha\beta} - l^{-2}g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{7}$$

$$\Box\Phi = \xi R\,\Phi + \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}, \tag{8}$$

where the energy-momentum tensor is defined by

$$T_{\alpha\beta} = \nabla_\alpha\Phi\nabla_\beta\Phi - g_{\alpha\beta}\left(\frac{1}{2}\nabla_\sigma\Phi\nabla^\sigma\Phi + U(\Phi)\right) + \xi\left(g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta + G_{\alpha\beta}\right)\Phi^2. \tag{9}$$

A distinctive feature of an AdS wave lies in the fact that its Einstein tensor has the following structure

$$G_{\alpha\beta} - l^{-2}g_{\alpha\beta} \propto k_\alpha k_\beta, \tag{10}$$

which in turn implies that any self-gravitating source supporting the wave in the presence of the negative cosmological constant must behave effectively as a pure radiation field [19]. As

a direct consequence, in the coordinates of metric (1), the only component of the Einstein equations with a nonvanishing left-hand side is the one along uu. Thus, the remaining Einstein equations reduce to the vanishing of the related energy-momentum tensor components. In what follows, we refer to these last conditions as the *pure radiation constraints*.

In the present case, we assume that the null Killing field $k^\mu\partial_\mu = \partial_v$ is also a symmetry of the scalar field, i.e. $\Phi = \Phi(u,y)$. The independent field equations on the AdS wave background (1) are written using the following combinations: the equation along the $uu-$component is given by

$$\begin{aligned} G_{uu} - l^{-2}g_{uu} - \kappa(T_{uu} - FT_{uv}) &= \frac{1}{2}(1-\kappa\xi\Phi^2)y\partial_y\left(\frac{1}{y}\partial_y F\right) \\ &\quad - \kappa\xi\left(\frac{1}{2}\partial_y\Phi^2\partial_y F - \partial^2_{uu}\Phi^2\right) - \kappa(\partial_u\Phi)^2 = 0, \end{aligned} \tag{11}$$

while the pure radiation constrains read

$$T_{uy} = \partial_u\Phi\partial_y\Phi - \frac{\xi}{y}\partial_y\left(y\partial_u\Phi^2\right) = 0, \tag{12a}$$

$$T_{yy} + T_{uv} = (\partial_y\Phi)^2 - \frac{\xi}{y^2}\partial_y\left(y^2\partial_y\Phi^2\right) = 0, \tag{12b}$$

$$T_{yy} = \frac{1}{2}(\partial_y\Phi)^2 - \frac{2\xi}{y}\partial_y\Phi^2 - \frac{1}{y^2}\left[l^2U(\Phi) - \xi\Phi^2\right] = 0. \tag{12c}$$

The nonlinear Klein-Gordon equation (8) reduces in this case to

$$\frac{y^3}{l^2}\partial_y\left(\frac{1}{y}\partial_y\Phi\right) = -\frac{6\xi}{l^2}\Phi + \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}. \tag{13}$$

Note that since the conservation equations of the energy-momentum tensor only involve the components given by Eqs. (12), the fulfillment of the pure radiation constraints guarantee that the equation (13) is automatically satisfied.

¿From these equations, it is easy to show the necessity of introducing a matter source. Indeed in the vacuum case, i.e. $\Phi = 0$, the resulting structural function would be given by

$$F(u,y) = F_2(u)y^2 - F_0(u), \tag{14}$$

and using the coordinate transformation (3), one can fix $F = 0$, and obtaining the metric of AdS.

III. THE SELF-INTERACTION POTENTIAL

Here, we explore the possibility of having a self-interacting nonminimally coupled ($\xi \neq 0$) scalar field acting as a source of an AdS wave background. As we shall see, the pure radiation constraints single out the form of the self-interaction potential as it was also the case in the *pp* wave context [17]. In order to derive the allowed potential, it is useful to redefine the scalar field as follows [2]

$$\Phi = \frac{1}{\sigma^{2\xi/(1-4\xi)}}. \tag{15}$$

[2] Clearly, the value $\xi = 1/4$ deserves a different analysis which is done at the end of this section.

With this redefinition the pure radiation constraints (12a) and (12b) are rewritten as

$$\partial_y \left(y\partial_u\sigma\right) \;=\; 0, \tag{16a}$$

$$\partial_y \left(y^2\partial_y\sigma\right) \;=\; 0, \tag{16b}$$

whose integration yields to

$$\sigma(u,y) = \frac{l}{y}\left(\sqrt{\lambda}y + \bar{f}(u)\right), \tag{17}$$

where λ is a positive constant and $\bar{f}$ is a general function of the retarded time. Using the expressions (15) and (17) it is possible to rewrite the remaining pure radiation constraint (12c) only in terms of the scalar field. This procedure fixes the self-interaction potential to be given by

$$U_\xi(\Phi) = \frac{2\xi\Phi^2}{(1-4\xi)^2}\left(\xi\lambda\Phi^{(1-4\xi)/\xi} - \frac{16}{l}\xi(\xi-1/8)\sqrt{\lambda}\Phi^{(1-4\xi)/(2\xi)} + \frac{24}{l^2}(\xi-1/8)(\xi-1/6)\right). \tag{18}$$

Various comments can be made concerning the structure of this potential. Firstly, for the three-dimensional conformal coupling $\xi = 1/8$, this potential reduces to the conformal one in three dimensions. Secondly, at the vanishing cosmological constant limit ($l \to \infty$), we recover the potential allowing a self-interacting scalar field to be nonminimally coupled to a *pp* wave background [17]. Finally, it is intriguing that this potential belongs to the same family of potentials arising in the context of scalar fields nonminimally coupled to special geometries without inducing backreaction (the static BTZ black hole [20], flat space [21], and the generalized (A)dS spacetimes [22]). Instead, all them has the common feature that they allow the existence of nontrivial solutions with a vanishing energy-momentum tensor called “stealth” configurations.

We would like to remark that the scalar field configurations given by Eqs. (15) and (17) solve the nonlinear Klein-Gordon equation (13) with a self-interaction potential given by Eq. (18) on any AdS wave background. In other words, this means that the scalar field and the allowed potential are not sensitive to the structural function F of the metric (1).

We now derive the self-interaction allowed by the nonminimal coupling value $\xi = 1/4$. In this case we redefine the scalar field as

$$\Phi = \frac{1}{\sqrt{\kappa}}\mathrm{e}^{\sigma}, \tag{19}$$

and the pure radiation constraints (12a) and (12b) reduce again to the equations (16). Hence, we conclude that the solution is given by expression (17), while the remaining pure radiation constraint (12c) gives rise to the following potential

$$U_{1/4}(\Phi) = \frac{\Phi^2}{4l^2}\left\{\left[2\ln\left(\sqrt{\kappa}\Phi\right) - l\sqrt{\lambda} + 1\right]^2 - 1\right\}. \tag{20}$$

For any value of the nonminimal coupling parameter, we have derived the allowed potential. In the next section, we explore the existence of free configurations which are obtained by imposing some appropriated restrictions on the potential (18).

IV. FREE SCALAR FIELDS

As shown before, the radiative constraints single out the form of the potential. For a generic value of the nonminimal coupling parameter and for a vanishing coupling constant $\lambda = 0$, the potential (18) becomes a mass term and hence, the scalar fields can be interpreted as free massive (or massless) fields. This argument is not valid for $\xi = 1/4$ since in this case the corresponding potential (20) does not allow the existence of free fields.

A. Massive cases

For a zero coupling constant $\lambda = 0$, the potential (18) reduces to a mass term given by

$$U(\Phi) = \frac{1}{2}{m_\xi}^2\Phi^2, \tag{21}$$

with a mass parameterized in terms of the nonminimal coupling parameter as

$${m_\xi}^2 = \frac{6\xi(\xi - 1/8)(\xi - 1/6)}{l^2(\xi - 1/4)^2}. \tag{22}$$

It is easy to see that in the minimal case $\xi = 0$, and for the three-dimensional (resp. four-dimensional) conformal coupling parameter, $\xi = 1/8$ (resp. $\xi = 1/6$), this mass vanishes and their related massless configurations will be analyzed in the next subsection. On the other hand, the above mass generated by the nonminimal coupling allows the existence of tachyonic solutions for negative values of the nonminimal coupling parameter and for $1/8 < \xi < 1/6$, as it is shown in FIG. 1.

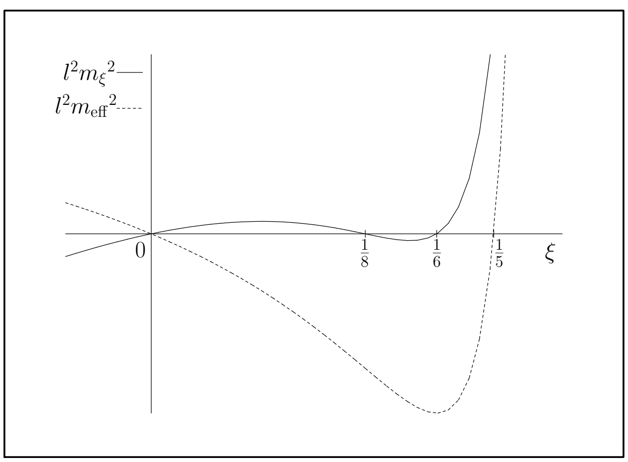

FIG. 1: The solid graph shows the dependence of the square of the scalar field mass ${m_\xi}^2$ on the nonminimal coupling parameter ξ, as fixed by the pure radiation constraints. The dotted graph corresponds to the dependence of the square of the effective mass ${m_{\text{eff}}}^2$, obtained when the contribution of the curvature is taken into account, see Eq. (38). The graphs are valid for $\xi \neq 1/4$. The dependence for $\xi > 1/4$ is not shown in the graph but it is positive definite in both cases.

For the nonminimal couplings associated with a nonvanishing mass the corresponding massive scalar field satisfying the pure radiation constraints is given by

$$\Phi(u,y)=\frac{1}{\sqrt{\kappa}}\left(\frac{y}{lf}\right)^{2\xi/(1-4\xi)}, \tag{23}$$

where we have rescaled appropriately the retarded-time-dependent function f to be dimensionless. Next, in order to determine the AdS wave supported by this massive field, we need to solve the Einstein equation (11). In this case, this equation can be rewritten as the following hypergeometric differential equation

$$x(x-1)\partial^2_{xx}H+\frac{x+2\xi-1}{2\xi}\partial_x H+\frac{1-4\xi}{2\xi}H=0, \tag{24}$$

after making the transformations

$$x = \xi\left(\frac{y}{lf}\right)^{4\xi/(1-4\xi)}, \tag{25a}$$

$$H(u,x) = \frac{l^2f^2}{y^2}F+l^2f\frac{\mathrm{d}^2f}{\mathrm{d}u^2}. \tag{25b}$$

The general solution of equation (24) reads

$$H(u,x)=F_1(u)\ {}_2\tilde{F}_1\left(1,\frac{1-4\xi}{2\xi};\frac{1-2\xi}{2\xi};x\right)+F_0(u)\left(\frac{x}{\xi}\right)^{(4\xi-1)/(2\xi)}, \tag{26}$$

where F_0 and F_1 are two undetermined functions of the retarded time and ${}_2\tilde{F}_1$ stands for the hypergeometric function [23]. In terms of the original variables it is possible to apply the coordinate transformation (3) which is equivalent to fix $F_0=0$ and $f=1$. Hence, the AdS waves supported by massive scalar fields, with a mass (22) generated by the nonminimal coupling, are given by

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\ {}_2\tilde{F}_1\left(1,\frac{1-4\xi}{2\xi};\frac{1-2\xi}{2\xi};\kappa\xi\Phi^2\right)\frac{y^2}{l^2}du^2-2dudv+dy^2\right], \tag{27a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}\left(\frac{y}{l}\right)^{2\xi/(1-4\xi)}, \tag{27b}$$

where the function F_1 has been properly rescaled.

Note that the above class of AdS wave configurations is not well-defined for the nonminimal coupling values

$$\xi_n=\frac{1}{2(1-n)},\qquad n=0,1,2,\ldots, \tag{28}$$

since the representation of the hypergeometric function in terms of the Gauss series is singular for these couplings. The value $n=1$ which corresponds to an infinite nonminimal coupling parameter (or equivalently to an infinite mass) is excluded in what follows. For the remaining values $n=0,2,3\ldots$, the mass is given by

$${m_n}^2=\frac{(n+2)(n+3)}{l^2(n+1)^2(1-n)}, \tag{29}$$

and the associated solution of Eq. (24) is given by

$$H_n(u,x) = \left\{ F_1(u) \left[\ln\left(1 - \frac{1}{x}\right) + \sum_{l=1}^{n+1} \frac{1}{l x^l} \right] + F_0(u) \right\} \left(\frac{x}{\xi_n}\right)^{n+1}. \tag{30}$$

Using again the coordinate transformation (3) to eliminate the functions F_0 and f, we conclude that for the nonminimal coupling values $\xi_n = 1/[2(1-n)]$ with $n = 0, 2, 3\ldots$, the solution reads

$$ds^2 = \frac{l^2}{y^2} \left\{ -F_1(u) \left[\ln\left(1 - \frac{1}{\kappa \xi_n \Phi^2}\right) + \sum_{l=1}^{n+1} \frac{1}{l(\kappa \xi_n \Phi^2)^l} \right] du^2 - 2du dv + dy^2 \right\}, \tag{31a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}} \left(\frac{l}{y}\right)^{1/(n+1)}. \tag{31b}$$

B. Massless cases

The simplest massless configuration is the minimal one. In fact for $\xi = 0$, it follows from the pure radiation constraints (12b) and (12c), that the theory can not accommodate a potential. Moreover, the corresponding free massless scalar field only depends arbitrarily on the retarded time, $\Phi = \Phi(u)$. In this case, the remaining independent Einstein equation (11) reduces to the three-dimensional inhomogeneous Siklos equation [8] for which the inhomogeneity is dictated by the scalar source,

$$\frac{1}{2} y \partial_y \left(\frac{1}{y} \partial_y F\right) = \kappa \left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2. \tag{32}$$

This equation is easily integrated as

$$F(u,y) = \kappa \left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 \left[\ln\left(\frac{y}{l}\right) + \bar{F}_2(u)\right] y^2 + F_0(u), \tag{33}$$

where F_0 and $\bar{F}_2$ denote two undetermined functions of the retarded time. We can now use a special version of the coordinate change (3) to eliminate the pure quadratic and zeroth order dependence on the wave-front coordinate y. In order to do that we impose to the function f appearing in the coordinate change to satisfy the equation (5) with F_2 given by

$$F_2 = \kappa \left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 \frac{(\bar{F}_2 + \ln f)}{f^4}. \tag{34}$$

This choice is motivated by the fact that, after rescaling the wave-front coordinate, an additional quadratic term appears arising from the logarithmic function. By doing that, we yield to the following minimally coupled field content

$$ds^2 = \frac{l^2}{y^2} \left[-\kappa \left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 \ln\left(\frac{y}{l}\right) y^2 du^2 - 2du dv + dy^2 \right]. \tag{35a}$$

$$\Phi = \Phi(u). \tag{35b}$$

Thus, in the minimal case, the scalar field has a wavy behavior which means that it allows an arbitrary profile in term of the retarded time, and this profile fully determines the profile of the AdS wave. Note that this property is also valid in the case of a *pp* wave configuration with a minimally coupled scalar field acting as source [17]. The introduction of a nonminimal coupling induces drastic changes in comparison with the conclusions of Ref. [17] regarding *pp* waves supported by free massless fields. For example, the pure radiation constraints for the AdS wave geometry are more restrictive than those for the *pp* wave one since for this later there are no restrictions on the nonminimal coupling parameter [17]. Indeed, for the AdS waves the other massless configurations (apart from the minimal one) are obtained only for the nonminimal couplings $\xi = 1/8$ and $\xi = 1/6$, as it can be concluded from the vanishing of the mass expression (22). These two systems are already considered within the general solution (27), but they are more easily expressed after fixing the coupling in each case. For the conformal coupling $\xi = 1/8$, the solution is given by

$$ds^2 = \frac{l^2}{y^2}\left\{-F_1(u)\left[\ln\left(1-\frac{y}{8l}\right)+\frac{y}{8l}\right]du^2 - 2dudv + dy^2\right\}, \tag{36a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}\sqrt{\frac{y}{l}}, \tag{36b}$$

while for the conformal coupling in four dimensions, $\xi = 1/6$, the solution becomes

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\ln\left(1-\frac{y^2}{6l^2}\right)du^2 - 2dudv + dy^2\right], \tag{37a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}\frac{y}{l}. \tag{37b}$$

In contrast with the minimally coupled massless configuration (35), the massless sources (36) and (37) have no wavy behavior. The reason is due to the presence of the cosmological constant which brings through the curvature an effective mass for any nonminimal coupling ($\xi \neq 0$) as it can be seen from the Klein-Gordon equation (13). This situation does not occur for the massless scalar fields nonminimally coupled to *pp* waves [17] since in this case the scalar curvature is identically zero.

This last observation motivates the following question: is there exists some specific values of the nonminimal coupling parameter for which the generated mass (22) compensates exactly the contribution of the cosmological constant to the effective mass? In this case, this would imply that the resulting configuration is a truly free massless field. In order to give an answer to this question, we first define the square of the effective mass as the sum of the generated mass (22) together with the contribution of the scalar curvature,

$${m_{\text{eff}}}^2 = \xi R + {m_\xi}^2 = \frac{5\xi(\xi-1/5)}{4l^2(\xi-1/4)^2}. \tag{38}$$

It is clear from this expression that for $\xi = 1/5$, which incidentally corresponds to the six-dimensional conformal coupling, the effective mass is zero. This in turn means that for $\xi = 1/5$, the pure radiation constraints allow massive fields which behave effectively as massless scalar fields. The corresponding fields also belong to the class described by Eq. (27)

and are expressed as

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\text{arctanh}\left(\frac{y^2}{\sqrt{5}l^2}\right)du^2 - 2dudv + dy^2\right], \tag{39a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}\frac{y^2}{l^2}. \tag{39b}$$

As it was expected, it is easy to see that the above scalar field satisfies the massless Klein-Gordon equation

$$\Box\Phi = 0, \tag{40}$$

and this solution is equivalent to the minimal one (35) in the sense that both describe genuinely massless fields on the curved AdS-wave background.

As a final remark, we note that configurations that behave effectively as tachyonic ones exist for values of the nonminimal parameter between zero and the other effectively-massless coupling $\xi = 1/5$, see FIG. 1.

V. SELF-INTERACTING SCALAR FIELDS

In this section we characterize the AdS waves supported by self-interacting nonminimally coupled scalar fields. As it has been shown in Sec. III, the pure radiation constraints fix the form of the potential which depends on a unique coupling constant λ. The properly self-interacting cases correspond to consider nonvanishing values of this coupling constant. In this case, the scalar source is described by Eqs. (15) and (17) as

$$\Phi(u,y) = \left[\frac{l}{y}\left(\sqrt{\lambda}\,y + \bar{f}(u)\right)\right]^{-2\xi/(1-4\xi)}. \tag{41}$$

Redefining the structural function by

$$H = F + \frac{1}{\bar{f}}\frac{\mathrm{d}^2\bar{f}}{\mathrm{d}u^2}y^2, \tag{42}$$

the equation (11) which determines the AdS wave background can be reduced to the following exact form

$$\partial_y\left[\frac{1}{y}\left(1 - \frac{\kappa\xi}{\left[\frac{l}{y}\left(\sqrt{\lambda}y + \bar{f}(u)\right)\right]^{4\xi/(1-4\xi)}}\right)\partial_y H\right] = 0. \tag{43}$$

This equation allows a first integral, and hence the function F can be determined in general in quadratures. The involved analytical dependence can be expressed in terms of standard functions just for special values of the nonminimal coupling parameter ξ. Additionally, its general behavior can change for critical values of the coupling constant λ. In order to exhibit these features in a concrete example, we analyze in details the self-interacting AdS wave configuration for the conformal coupling $\xi = 1/8$.

For $\xi = 1/8$, the self-interaction potential (18) reduces to the conformal potential in $2+1$ dimensions

$$U_{1/8}(\Phi) = \frac{\lambda}{8}\Phi^6. \tag{44}$$

For a coupling constant $\lambda \neq (\kappa/8l)^2$, it is possible to redefine the dependence on the retarded time as $\bar{f} = (\kappa - 8l\sqrt{\lambda})f$, where f is a dimensionless function. In this case, the solution of Eq. (43) turns out to be

$$H(u,y) = F_1(u)\left[\frac{\sqrt{\lambda}y^2}{16l\kappa f^2} + \frac{y}{8lf} + \ln\left(1 - \frac{y}{8lf}\right)\right] + F_0(u). \tag{45}$$

Returning to the original structural function by means of Eq. (42), and performing the coordinate transformation (3) it is possible to set $F_0 = 0$ and $f = 1$. Hence, for a conformal coupling $\xi = 1/8$ with a generic coupling constant $\lambda \neq (\kappa/8l)^2$, the related AdS wave configuration is given by

$$ds^2 = \frac{l^2}{y^2}\left\{-F_1(u)\left[\frac{\sqrt{\lambda}y^2}{16l\kappa} + \frac{y}{8l} + \ln\left(1 - \frac{y}{8l}\right)\right]du^2 - 2dudv + dy^2\right\}, \tag{46a}$$

$$\Phi = \frac{1}{\sqrt{\kappa}}\sqrt{\frac{y}{l}}\left[\frac{8l\sqrt{\lambda}}{\kappa}\left(\frac{y}{8l} - 1\right) + 1\right]^{-1/2}, \tag{46b}$$

where F_1 has been rescaled conveniently. It is interesting to note that for $\lambda = 0$, the free massless configuration (36) is straightforwardly obtained. In fact, both gravitational fields are related by a generalized Kerr-Schild transformation

$$g_{\mu\nu}^{\lambda} = g_{\mu\nu}^{\lambda=0} - \sqrt{\lambda}\frac{F_1 y^4}{16l^3\kappa}k_\mu k_\nu. \tag{47}$$

In other words, this means that the self-interacting configuration can be described just as a gravitational wave propagating on the background of the free massless configuration: an exact gravitational wave on another exact gravitational wave. Since $g_{\mu\nu}^{\lambda=0}$ is already a Kerr-Schild transformation from AdS space, expression (47) coincides with what is known in the literature as a double Kerr-Schild transformation [19].

More intriguing is the fact that the manifestation of the self-interaction arises exactly in a quadratic dependence on the wave-front coordinate y. Until now, it has always be possible and useful to eliminate locally these contributions by mean of the coordinate transformation (3). However, in the self-interacting case there is no isolated quadratic coefficient, since the profile function F_1 is a common coefficient to the whole wave-front dependence of the structural function F as it can be seen from Eq. (46). Thus, it is not judicious to eliminate this term using the previous arguments, and it seems that the nonvanishing coupling constant becomes an obstruction to this local mechanism.

We now analyze the case for which the coupling constant takes the special value $\lambda = (\kappa/8l)^2$. For a such value, we rewrite the retarded time dependent function as $\bar{f} = \kappa f/8$, where f is again a dimensionless function. The Eq. (43) is now solved by

$$H(u,y) = F_1(u)\left(\frac{y^3}{3l^3f^3} + \frac{y^2}{2l^2f^2}\right) + F_0(u). \tag{48}$$

By fixing $f = 1$ and $F_0 = 0$ as usual, we end with the following AdS wave configuration valid for the conformal coupling $\xi = 1/8$ and the special coupling constant $\lambda = (\kappa/8l)^2$,

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\left(\frac{y^3}{3l^3} + \frac{y^2}{2l^2}\right)du^2 - 2dudv + dy^2\right], \tag{49a}$$

$$\Phi = \sqrt{\frac{8}{\kappa}\frac{y}{y+l}}, \tag{49b}$$

where the function F_1 has been rescaled. The above configuration presents an outstanding non-perturbative feature since the strong and weak regimes of the source have similar behavior. This can be shown by first establishing that the fields defined by

$$\hat{\Phi} = \frac{8}{\kappa}\frac{1}{\Phi}, \quad \hat{ds}^2 = \left(\sqrt{\frac{\kappa}{8}}\,\Phi\right)^4 ds^2, \tag{50}$$

are also solutions of the field equations. Indeed, modulo the coordinate transformation

$$(u, v, y) \mapsto \left(\hat{u} = u, \hat{v} = v + \frac{1}{12}\int \mathrm{d}u F_1, \hat{y} = -y - l\right), \tag{51}$$

the new fields can be written as

$$\hat{ds}^2 = \frac{l^2}{\hat{y}^2}\left[F_1(\hat{u})\left(\frac{\hat{y}^3}{3l^3} + \frac{\hat{y}^2}{2l^2}\right)d\hat{u}^2 - 2d\hat{u}d\hat{v} + d\hat{y}^2\right], \tag{52a}$$

$$\hat{\Phi} = \sqrt{\frac{8}{\kappa}\frac{\hat{y}}{\hat{y}+l}}. \tag{52b}$$

These expressions describe the same solution than the one given by Eqs. (49) but with a reflected profile $\hat{F}_1(\hat{u}) = -F_1(\hat{u})$. Since the new and old scalar fields are inversely proportional, one can conclude a kind of strong-weak duality where both regimes are diffeomorphic to each other. Moreover, these two regimes can only be differentiated by the reflected profiles of their corresponding AdS waves.

Finally, much of the previous analysis can be extended to other nonminimal couplings. For example for the family of special values $\xi = m/[4(m+1)]$, $m \in \mathbb{Z} \setminus \{-1\}$, the power $4\xi/(1-4\xi)$ which appears in Eq. (43) takes the integer value m. In these cases the wave-front dependence is similar to the dependence obtained in the conformal case (45). That is, the solution H of the equation (42) can be written as a common retarded-time-dependent coefficient multiplying a quadratic term on the wave-front coordinate, a linear one, and several logarithmic terms for which the coefficients depend on the roots of a polynomial of order m.

VI. ADS WAVES *VS PP* WAVES SOURCES: A CORRESPONDENCE

As shown in Sec. III, the AdS wave source, that means the scalar field Φ together with its allowed potential $U(\Phi)$, is completely determined by resolving the pure radiation constraints. These later which are three of the four independent Einstein equations do not involve the structural metric function F as it can be verified from their definitions (12). In the case of a scalar field nonminimally coupled to a *pp* wave, a similar situation occurs. In the present section, we show that this is not merely a coincidence and, moreover the analogy can be extended by establishing a correspondence between the AdS wave and the *pp* wave sources. In particular, as shown below, starting from a *pp* wave configuration one can derive the AdS wave source and *vice et versa*.

In order to make the discussion self-contained we first give a short review about scalar fields nonminimally coupled to a *pp* wave [17],

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = -\bar{F}(u, y)du^2 - 2dudv + dy^2, \tag{53}$$

for which the involved field equations are given by

$$\bar{G}_{\alpha\beta} = \kappa \bar{T}_{\alpha\beta}, \tag{54}$$

$$\bar{\Box}\bar{\Phi} = \xi \bar{R}\bar{\Phi} + \frac{\mathrm{d}\bar{U}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}. \tag{55}$$

Since the Einstein tensor of a *pp* wave geometry (53) satisfies

$$\bar{G}_{\alpha\beta} \propto k_\alpha k_\beta, \tag{56}$$

the coupling of any matter source to a *pp* wave also imposes pure radiation constraints, which in the present case are given by

$$\begin{aligned}
\bar{T}_{uy} &= \partial_u \bar{\Phi} \partial_y \bar{\Phi} - \xi \partial_{uy} \bar{\Phi}^2 = 0, && \text{(57a)}\\
\bar{T}_{yy} + \bar{T}_{uv} &= (\partial_y \bar{\Phi})^2 - \xi \partial_{yy} \bar{\Phi}^2 = 0, && \text{(57b)}\\
\bar{T}_{yy} &= \frac{1}{2}(\partial_y \bar{\Phi})^2 - \bar{U}(\bar{\Phi}) = 0. && \text{(57c)}
\end{aligned}$$

In Ref. [17], we showed that the scalar field is determined independently of the metric function $\bar{F}$ by solving the *pp* wave pure radiation constraints (57). In addition, these constraints restrict the self-interaction potential to be

$$\bar{U}(\bar{\Phi}) = \frac{2\xi^2\bar{\lambda}}{(1-4\xi)^2}\bar{\Phi}^{(1-2\xi)/\xi}, \tag{58}$$

where $\bar{\lambda}$ is a positive coupling constant.[3] To be complete, we mention that in the free case ($\bar{\lambda} = 0$), the scalar field is shown to be an arbitrary function of the retarded time $\bar{\Phi} = \bar{\Phi}(u)$, while in the self-interacting case ($\bar{\lambda} \neq 0$), it is given by

$$\bar{\Phi}(u, y) = \left(\sqrt{\bar{\lambda}}\, y + \bar{f}(u)\right)^{-2\xi/(1-4\xi)}, \tag{59}$$

where $\bar{f}$ is an arbitrary function of the retarded time. Finally, the *pp* wave profile $\bar{F}$ is determined by solving the remaining independent Einstein equation, $\bar{G}_{uu} = \kappa \bar{T}_{uu}$.

At this point we would like to stress that the self-interacting scalar fields supporting the AdS wave (41) and the *pp* wave (59) are functionally related as

$$\Phi = \Omega^{-2\xi/(1-4\xi)}\bar{\Phi}, \tag{60}$$

where we have assumed that $\lambda = \bar{\lambda}$. In this relation, $\Omega = l/y$, and corresponds precisely to the conformal factor that allows to define the AdS wave metric (1) as a conformal transformation of some *pp* wave background.

Inspired by the previous functional relation, we explore the possibility of establishing a complete correspondence between the AdS wave scalar source and the *pp* wave one. More precisely, we shall prove the following result: starting from the *pp* wave scalar source, i.e. the scalar field (59) and the allowed potential (58), we are able to generate the AdS wave

[3] The coupling constant used in Ref. [17] expressed in terms of the one used here is $\bar{\lambda}/4$.

scalar source described by Eqs. (41) and (18), and *vice et versa*. This is done by assuming that the scalar fields are conformally related with a conformal factor given by $\Omega = l/y$ but without fixing the weight s.

In the first part of the next subsection, we show that for a weight $s = -2\xi/(1-4\xi)$ and for the allowed self-interaction potentials (18) and (58) with same coupling constant, the pure radiation constraints of the AdS wave (12) are conformally related to those of the *pp* wave (57). Note that this correspondence is strictly realized between the sources and not between the involved structural functions, F and $\bar{F}$. However, in the particular situation of retarded-time-dependent sources, a relation between these structural functions can be derived. The details are given in the second part of the next subsection. Finally, the last subsection is dedicated to configurations that do not depend on the retarded time for which a more general form of the correspondence can be derived.

A. Off-shell correspondence

1. The sources

In this subsection, we establish an off-shell correspondence between the *pp* wave and the AdS wave matter sources. By off-shell, we mean that the correspondence is realized without using the explicit form of the scalar field solutions. The expression (60) suggests to consider a conformal relation between the self-interacting scalar fields on both gravitational wave backgrounds as

$$\Phi = \left(\frac{l}{y}\right)^{s}\bar{\Phi}, \tag{61}$$

where the weight s will be fixed later. In this case, the AdS wave pure radiations constraints (12) can be expressed in terms of those of the *pp* wave (57) as follows

$$T_{uy} = \left(\frac{l}{y}\right)^{2s}\left(\bar{T}_{uy} - [s(1-4\xi)+2\xi]\,\frac{\partial_u\bar{\Phi}^2}{2y}\right), \tag{62a}$$

$$T_{yy}+T_{uv} = \left(\frac{l}{y}\right)^{2s}\left[\bar{T}_{yy}+\bar{T}_{uv} - [s(1-4\xi)+2\xi]\,y^{s-1}\partial_y\left(\frac{\bar{\Phi}^2}{y^s}\right)\right], \tag{62b}$$

$$T_{yy}+\frac{l^2}{y^2}\left[U(\Phi)-V_s(y,\Phi)\right] = \left(\frac{l}{y}\right)^{2s}\left(1+\frac{\sqrt{2}(s+4\xi)\bar{\Phi}}{y\left[\sqrt{\bar{U}(\bar{\Phi})}+\sqrt{\bar{T}_{yy}+\bar{U}(\bar{\Phi})}\right]}\right)\bar{T}_{yy}, \tag{62c}$$

where the function $V_s(y,\Phi)$ is defined by

$$V_s(y,\Phi) = \left(\frac{l}{y}\right)^{\frac{s(4\xi-1)-2\xi}{\xi}}\bar{U}(\Phi) + \left(\frac{l}{y}\right)^{\frac{s(4\xi-1)-2\xi}{2\xi}}\frac{(s+4\xi)}{l}\Phi\sqrt{2\bar{U}(\Phi)} + \left(s^2+8\xi s+2\xi\right)\frac{\Phi^2}{2l^2}, \tag{63}$$

and the functional dependence $\bar{U}$ is given by Eq. (58).

¿From the first two relations (62a) and (62b), it is simple to see that the involved pure radiations constraints of both backgrounds are conformally related if the weight is given by

$$s = -\frac{2\xi}{1-4\xi}. \tag{64}$$

In addition, for this value of the weight the function (63) looses its dependence on the wave-front coordinate y and becomes a local function of the scalar field expressed as

$$V_{-2\xi/(1-4\xi)}(y,\Phi)=\bar{U}(\Phi)-\frac{16\xi\,(\xi-1/8)}{(1-4\xi)l}\Phi\sqrt{2\bar{U}(\Phi)}+\frac{48\xi(\xi-1/8)(\xi-1/6)}{(1-4\xi)^2l^2}\Phi^2. \tag{65}$$

As it can be seen from the relation (62c), the remaining pure radiation constraints are also conformally related provided that the AdS wave potential $U(\Phi)$ is given by the expression (65). Note that this expression exactly corresponds to the potential allowed by the AdS wave source (18) for which the coupling constant is taken as $\lambda=\bar{\lambda}$.

In sum, the pure radiation constraints on both gravitational waves are conformally related with weight $s=-2\xi/(1-4\xi)$, when their respective potentials (58) and (18) are taken with the same coupling constants $\lambda=\bar{\lambda}$, i.e.

$$T_{uy}=\left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\bar{T}_{uy}, \tag{66a}$$

$$T_{yy}+T_{uv}=\left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\left(\bar{T}_{yy}+\bar{T}_{uv}\right), \tag{66b}$$

$$T_{yy}=\left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\left(1-\frac{16\sqrt{2}\xi(\xi-1/8)\bar{\Phi}}{(1-4\xi)y\left[\sqrt{\bar{U}(\bar{\Phi})}+\sqrt{\bar{T}_{yy}+\bar{U}(\bar{\Phi})}\right]}\right)\bar{T}_{yy}. \tag{66c}$$

Using these relations, it is possible to generate any AdS wave scalar source from the *pp* wave source through a conformal transformation with a particular weight depending on the nonminimal coupling parameter, and considering the same coupling constant in both cases $\lambda=\bar{\lambda}$. For example, for $\bar{\lambda}\neq 0$, the self-interacting scalar solutions supporting the *pp* waves also generate the self-interacting sources ($\lambda\neq 0$) which support the AdS waves. On the other hand, for $\bar{\lambda}=0$ the free massless *pp* wave scalar fields depend only on the retarded time and allow to generate the free massive and massless configurations ($\lambda=0$) supporting the AdS waves analyzed in Sec IV.

Finally, we note that similar conclusions can be obtained by considering the scalar wave equations. Using the conformal relation (61) with the weight (64) and assuming $\lambda=\bar{\lambda}$, the Klein-Gordon equations in both backgrounds are not conformally related in general. However, a conformal relation can be reached for the following combination involving the wave equations

$$\begin{aligned}2\xi\Phi\left(\Box\Phi-\xi R\Phi-\frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}\right)-g^{yy}T_{yy} &= \left(\frac{l}{y}\right)^{2(2\xi-1)/(1-4\xi)}\\ &\times\left[2\xi\bar{\Phi}\left(\bar{\Box}\bar{\Phi}-\xi\bar{R}\bar{\Phi}-\frac{\mathrm{d}\bar{U}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}\right)-\bar{g}^{yy}\bar{T}_{yy}\right].\end{aligned} \tag{67}$$

It is clear from this expression that any solution of the wave equation in one background satisfying also the pure radiation constraints must be necessarily a solution of the wave equation in the other background. A particularity obviously occurs for the conformal coupling $\xi=1/8$, since in this case one can use the relation (66c) to eliminate the energy-momentum components from the above combinations, yielding to the standard conformal correspondence between the wave equations in this case. We never have seen a relation like the above

in the literature, it has sense only for strictly nonminimal couplings ($\xi \neq 0$) and gravitational waves backgrounds, but it would be very interesting if there exist other geometries allowing its existence.

2. The backgrounds

In the above treatment we have established a correspondence between the scalar sources supporting a *pp* wave and an AdS wave. Here we shall show that in the case of scalar fields depending on the retarded time it is also possible to build a relation between both gravitational waves. Up to now in this section, the only unexplored component of the Einstein equations has been the *uu* one, since it does not participate in the pure radiations constraints. We shall use it now, due to it is the only component containing information on the metric structural functions.

Using the conformal relation (61) and eliminating the second derivatives of the scalar field with respect to the retarded time from the *uu* components of the Einstein equations in both backgrounds the following relation is obtained

$$E_{uu} - FE_{uv} - \frac{y}{2}\partial_y\left(\left(1-\kappa\xi\Phi^2\right)\frac{\partial_y F}{y}\right) = \left(\frac{l}{y}\right)^{2s}\left(\bar{E}_{uu} - \bar{F}\bar{E}_{uv} - \frac{1}{2}\partial_y\left[\left(1-\kappa\xi\bar{\Phi}^2\right)\partial_y\bar{F}\right]\right), \tag{68}$$

where $E_{\alpha\beta}$ and $\bar{E}_{\alpha\beta}$ denote the components of the Einstein equations for the AdS and *pp* wave cases, respectively, i.e. they are defined by

$$E_{\alpha\beta} = G_{\alpha\beta} - l^{-2}g_{\alpha\beta} - \kappa T_{\alpha\beta}, \tag{69}$$

$$\bar{E}_{\alpha\beta} = \bar{G}_{\alpha\beta} - \kappa\bar{T}_{\alpha\beta}. \tag{70}$$

As consequence of the identity (68) a *pp* wave solution ($\bar{E}_{\alpha\beta} = 0$) implies the existence of an AdS wave solution ($E_{\alpha\beta} = 0$), and viceversa, iff both structural functions satisfy a differential equation which allows to determine one solution in terms of the other. For example, for obtaining the AdS wave solution the differential equation is given by

$$y\partial_y\left\{\left[1-\kappa\xi\left(\frac{y}{l}\right)^{4\xi/(1-4\xi)}\bar{\Phi}^2\right]\frac{\partial_y F}{y}\right\} = \left(\frac{y}{l}\right)^{4\xi/(1-4\xi)}\partial_y\left[\left(1-\kappa\xi\bar{\Phi}^2\right)\partial_y\bar{F}\right], \tag{71}$$

where we are additionally using that the remaining Einstein equations imply that $s = -2\xi/(1-4\xi)$ as was shown previously. Finally, the above equation allows to determine in quadratures the AdS wave structural function F in terms of the *pp* wave solution

$$F(u,y) = \int\frac{\mathrm{d}y\, y}{\left(1-\kappa\xi\left(y/l\right)^{4\xi/(1-4\xi)}\bar{\Phi}^2\right)}\left[F_1(u) + \int\frac{\mathrm{d}y}{l}\left(\frac{y}{l}\right)^{(8\xi-1)/(1-4\xi)}\partial_y\left[\left(1-\kappa\xi\bar{\Phi}^2\right)\partial_y\bar{F}\right]\right] + F_0(u). \tag{72}$$

In the case of scalar configurations independent of the retarded time the relation (68) can not be achieved. It appears that there is no relation between the backgrounds in this case.

B. On-shell correspondence

In the AdS wave case and for a generic nonminimal coupling parameter ($\xi \neq 0$), the retarded-time-dependent integration functions of the scalar field can always be put equal to some non zero constant by an appropriate coordinate transformation. In the case of the *pp* wave configurations, the situation is different. Indeed, as shown in Ref. [17], the arbitrary retarded time dependent functions can only be removed for a self-interacting scalar field. In this case, a suitable shift in the wave-front coordinate y allows to remove the integration function $\bar{f}$ from (59) leading to the following "physical" solution

$$\bar{\Phi} = \left(\sqrt{\bar{\lambda}}\, y\right)^{-2\xi/(1-4\xi)}. \tag{73}$$

In view of these remarks, it is not clear that the correspondence reported previously is still valid once the undetermined integration functions have been removed. In fact, we shall see that the two systems, namely the self-interacting scalar fields nonminimally coupled to a *pp* wave and the nonminimal configurations supporting the AdS wave (once the integration functions have been removed in each case) can also be linked through a more general correspondence.

We start by looking for a more general relation between the scalar fields,

$$\Phi = H(\Omega)^{-2\xi/(1-4\xi)}\, \bar{\Phi}, \tag{74}$$

where H is a function of the conformal factor $\Omega = l/y$ and $\bar{\Phi}$ is the physical *pp* wave scalar field (73). In this case, the AdS wave pure radiations constraints (12) reduce to

$$T_{uy} = 0, \tag{75a}$$

$$T_{uv} + T_{yy} = \frac{4\xi^2\Omega^5\Phi^2}{l^2(1-4\xi)H}\, \frac{\mathrm{d}^2}{\mathrm{d}\Omega^2}\left(\frac{H}{\Omega}\right), \tag{75b}$$

$$\begin{aligned} T_{yy} = {} & -\Omega^2 U(\Phi) + \frac{\xi\Phi^2\Omega^2}{H^2(4\xi-1)^2\, l^2} \\ & \times \left[2\xi\Omega^2\left(\frac{\mathrm{d}H}{\mathrm{d}\Omega}\right)^2 + 4\xi(8\xi-3)\Omega H\frac{\mathrm{d}H}{\mathrm{d}\Omega} + (1+2\xi-16\xi^2)H^2\right], \end{aligned} \tag{75c}$$

where in order to derive these relations, we have make use of the explicit form of the scalar field solution nonminimally coupled to a *pp* wave, (73).

The vanishing of the second equation (75b) implies that the function H is given by

$$H(\Omega) = \alpha\Omega + \beta\Omega^2, \tag{76}$$

where α and β are two arbitrary constants. Combining this expression with the explicit form of the scalar field solution (73) on a *pp* wave background, we find that the AdS wave scalar field can be written as

$$\Phi = \left[\frac{l}{y}\left(\alpha\sqrt{\bar{\lambda}}\, y + \beta\sqrt{\bar{\lambda}} l\right)\right]^{-2\xi/(1-4\xi)} = \left[\sqrt{\bar{\lambda}} l\left(\alpha + \beta\Omega\right)\right]^{-2\xi/(1-4\xi)}. \tag{77}$$

This dependence is identical to the solution given in Eq. (41) where $\alpha^2\bar{\lambda}$ plays now the role of the coupling constant and the function depending on the retarded time is given by the constant $\beta\sqrt{\bar{\lambda}}l$. In fact, using the above expression in the vanishing of Eq. (75c) one recovers the self-interacting potential (18) allowing the AdS wave configurations, but with a coupling constant given now by $\lambda = \alpha^2\bar{\lambda}$. This reflects the fact that by construction, the equations determining the AdS wave source, namely the pure radiation constraints and the nonlinear Klein-Gordon equation, are invariant under the rescaling $\lambda \to \alpha^2\lambda$. This is a consequence of the fact that the scalar contribution to action (6) only change by a multiplicative constant under the above rescaling. As it was previously anticipated, the free constant β corresponds to the nonzero constant remaining in the scalar field once the retarded-time-dependent integration function is removed. It can be fixed appropriately using the transformations (3), as it has been explicitly shown in Sec. V.

For $\alpha \neq 0$ the present correspondence is analog to the correspondence established in the previous subsection, in the sense that it allows to relate the self-interacting configurations on a *pp* wave (i.e. $\bar{\lambda} \neq 0$) with the self-interacting configurations on an AdS wave (i.e. $\lambda = \alpha^2\bar{\lambda} \neq 0$). The basic difference lies in the fact that the integration functions depending on retarded time are appropriately fixed in each case. However, for $\alpha = 0$, a new link can be made. Indeed, in this case, the coupling constant in the AdS wave side vanishes ($\lambda = \alpha^2\bar{\lambda} = 0$) and as a consequence, the potential reduces to the mass term given by Eqs. (21) and (22). Hence, the free scalar fields on an AdS wave (27b) and the self-interacting scalar fields on a *pp* wave (73) are effectively related provided the constant β to be fixed by

$$\beta = \left(\sqrt{\bar{\lambda}}\,l\right)^{-1} \kappa^{(1-4\xi)/(4\xi)}.$$

In summary, we have generated the free massive and massless configurations ($\lambda = 0$) supporting an AdS wave from the self-interacting *pp* wave configurations ($\bar{\lambda} \neq 0$).

VII. CONCLUSIONS

In this paper, we have attacked the problem of the generation of exact gravitational waves propagating on AdS space. We have restricted our attention to the three-dimensional case for which a matter source must be necessarily introduced in order to support these AdS waves. The elaboration of this work has opened many interesting questions that go beyond the simple mathematical resolution.

The first interrogation may concern the choice of the matter source for these AdS waves. Indeed, due to the wave character of these gravitational fields, any matter supporting them must behave as a pure radiation field. This means that the only nonvanishing component of the energy momentum tensor must be the energy density along the retarded time. This last fact is certainly very restrictive concerning the possible choices of matter source. It is interesting to remark that a source given by a scalar field nonminimally coupled to the AdS waves, as we have considered here, does not yield to inconsistencies, quite the contrary provides many interesting curiosities that we have reported throughout this paper. Although it is also natural to ask about the motivation of considering such source instead of another plausible one, the appearance of these unexpected curiosities (essentially due to the inclusion of the nonminimal coupling) in some sense legitimate our choice. As the first curiosity, it is intriguing that the integration of the *pure radiation constraints*, which are three of the four independent Einstein equations, completely fixes the dependence of the scalar field as well

as singles out a unique self-interaction potential allowing the existence of AdS waves. In other words, this means that the self-interacting nature of the source is strictly determined from the requirement of being capable of generating AdS waves.

We showed that the resulting potential depends only on one coupling constant denoted by λ and presents various interesting characteristics. Its expression is given by a superposition of different powers of the scalar field whose exponents are expressed in terms of the nonminimal coupling parameter ξ. For the three-dimensional conformal coupling, i.e. $\xi = 1/8$, the parameterized expression of the potential reduces precisely to the three-dimensional conformal potential $U_{1/8}(\Phi) \propto \Phi^6$. This in turn implies that the matter source allowed by the AdS waves becomes conformally invariant. It is amusing to note that there is *à priori* no good reason for the potential to become the conformally invariant one, since the full system does not exhibit the conformal invariance. We would like to remark that this is not the first time that the above nonminimal-coupling related potential appears. In the limit of vanishing cosmological constant $l \to \infty$ the potential becomes the one allowing the existence of scalar-field generated *pp* waves derived in Ref. [17]. Additionally, it is a particular case of the family of potentials allowing the existence of gravitationally stealth configurations on special backgrounds as the static BTZ black hole, flat space, and (A)dS, see Refs. [20, 21, 22].

After determining the scalar source from the pure radiation constraints, the integration of the remaining Einstein equation permits to find the geometric background. The simplest case occurs for a vanishing coupling constant $\lambda = 0$, since the potential reduces to a pure mass term with the peculiarity that the corresponding mass ${m_\xi}^2$ is fixed in terms of the nonminimal coupling parameter with a mass scale determined from the cosmological constant. This last fact increases the interest on the nonminimal coupling parameter ξ whose range now allows the existence of several types of free sources. For example, for $1/8 < \xi < 1/6$ or $\xi < 0$, the AdS waves are generated by tachyonic configurations (${m_\xi}^2 < 0$). The waves can also be supported by massless fields in the cases of the minimal coupling $\xi = 0$, the three-dimensional conformal coupling $\xi = 1/8$, and the four-dimensional conformal coupling $\xi = 1/6$. However, the last two cases do not represent genuinely massless fields in their corresponding curved backgrounds. This is due to the fact that there is an additional contribution to their mass coming from de nonminimal coupling of the fields to the gravitational waves. Since the AdS waves has negative scalar curvature this contribution is tachyonic, as a consequence the fields acquire an effective mass given by ${m_{\text{eff}}}^2 = -\xi 6 l^{-2} + {m_\xi}^2$ on these backgrounds. The explicit dependence of the effective mass on the nonminimal coupling parameter implies that in the range $0 < \xi < 1/5$ the scalar fields are forced to behave effectively as tachyonic fields (${m_{\text{eff}}}^2 < 0$). As it can be anticipated from continuity, the genuinely massless states (${m_{\text{eff}}}^2 = 0$) are not only realized for the minimal coupling $\xi = 0$ but also for the nonminimal coupling value $\xi = 1/5$, which incidentally corresponds to the conformal coupling in $D = 6$. The value $\xi = 1/5$ is associated to a critical case in the sense that its allowed mass ${m_{1/5}}^2$ exactly compensates the contribution generated by the negative cosmological constant, this explain why the scalar field becomes a truly massless field in a curved background. It is true that the matter source is characterized by a unique parameter, namely the nonminimal coupling parameter, but once again there is no good reason for the existence of a critical value of this parameter that renders the scalar field effectively as a massless field. Additionally, the fact that the fine-tuning occurs for the conformal coupling of a higher dimension seems to indicate that the effect can be connected to some more symmetrical higher-dimensional physics. An interesting work will consist in exploring if this mass annihilating effect can be extended to higher-dimensional versions of the AdS waves

or to other geometries.

The self-interacting case, i.e. $\lambda \neq 0$, has been studied in details for the conformal coupling $\xi = 1/8$. Generically, the background have been shown to be a gravitational wave propagating on AdS space builds from the superposition of two contributions. The first one corresponds to the massless free field ($\lambda = 0$). The other contribution is associated with the self-interaction, since it presents a dependence on the coupling constant which goes as $\sim \sqrt{\lambda}$. As a consequence, the free configuration is consistently obtained from the self-interacting one in the limit $\lambda \to 0$. This double Kerr-Schild representation of the background starting from AdS space is outstanding, since in spite of the strong self-coupling of the field ($\propto \Phi^6$) its contribution as source is encoded in a very simple way. This simple connection between the free and the self-interacting cases occurs only if the coupling constant is different from the critical value $\lambda = (\kappa/8l)^2$. For this critical value, we have proved that a surprising non-perturbative effect occurs. We have established a kind of strong-weak duality in the sense that the field strengths Φ and $1/\Phi$ are diffeomorphic and the profile of their corresponding AdS waves only differs by a minus sign. In other words the strong regime is locally equivalent to the weak one just by reflecting the profile of the gravitational wave. These results for the conformal coupling can be generalized to the family of nonminimal coupling values $\xi = m/[4(m+1)]$, $m \in \mathbb{Z} \setminus \{-1\}$, and in the same line as in the conformal case, different critical behaviors can also appear.

Another intriguing fact concerns the relation that we have established with the problem of scalar fields nonminimally coupled to a *pp* wave. It is well-known that an AdS wave can also be viewed as a conformal transformation of a *pp* wave but there is no reason for their matter sources to be also in correspondence. Here, we have shown that the pure radiation constraints of both systems are conformally related when their scalar fields are also conformally related with a conformal weight fixed in terms of the nonminimal coupling parameter. This conformal correspondence occurs between the scalar sources (scalar fields and allowed potentials in each case) but the correspondence can not be extended to the involved backgrounds. In addition, we have shown that a particular combination of the Klein-Gordon equation together with the wave-front component of the energy-momentum tensor T_{yy} is in fact conformally invariant in the pass from one gravitational wave to the other. In the case of the conformal coupling this relation becomes the well-known conformal invariance of the conformal Klein-Gordon equation. It will be interesting to explore the transcendence of similar conclusions in higher dimensions.

Finally, we have considered the case for which the AdS waves are governed by topologically massive gravity with a negative cosmological constant. Unlike standard $2+1$ gravity, the waves are allowed in the vacuum case. The negative cosmological constant acts on the gravitational waves by reducing the value of their topological mass, i.e. if the theory has a topological mass μ the AdS waves have a physical topological mass $\mu_{\text{eff}} = \mu\sqrt{1-(l\mu)^{-2}}$. These vacuum configurations coincide with the ones derived in Ref. [16] from a correspondence with conformal gravity. For the critical values of the topological mass $\mu = \pm l^{-1}$, the corresponding AdS waves can not be interpreted as massive Klein-Gordon modes, which seems to indicate that in general the theory has no massive character in these limits. The above considerations have been extended to waves supported by free nonminimally coupled scalar fields. In the minimal case the gravitational wave is just a superposition of two contributions given by the vacuum AdS wave and by the AdS wave corresponding to Einstein gravity supported by a minimally coupled scalar field with an effective gravitational constant $\kappa_{\text{eff}} = \kappa/[1+(l\mu)^{-1}]$. We have also studied the zero topological mass limit of the

above nonvacuum configurations and showed that they become the AdS wave solutions of conformal gravity.

Acknowledgments

We thank D. Correa, C. Martínez, M. Ortaggio, R. Troncoso, and J. Zanelli for useful discussions and especially A. Garcia. This work is partially supported by grants 3020032, 1040921, 7040190, 1051064, 1051084, and 1060831 from FONDECYT, grants 38495E, 34222E, and CO1-41639 from CONACyT, and grant 2001-5-02-159 from CONICYT/CONACyT. Institutional support to the Centro de Estudios Científicos (CECS) from Empresas CMPC is gratefully acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

APPENDIX: ADS WAVES FOR TOPOLOGICALLY MASSIVE GRAVITY WITH A COSMOLOGICAL CONSTANT

In this Appendix, we analyze the AdS waves ruled by topologically massive gravity [24], when this theory is supplemented with a negative cosmological constant. The scalar configurations nonminimally coupled to the *pp* waves of this theory were studied in Refs. [15, 16, 17, 18] using different perspectives. The topologically massive gravity modifies the Einstein equations (7) by the addition of the Cotton tensor. However, for the AdS wave metric (1) the corresponding left hand side is again constrained to satisfy

$$\frac{1}{\mu}C_{\alpha\beta} + G_{\alpha\beta} - l^{-2}g_{\alpha\beta} \propto k_\alpha k_\beta, \tag{A.1}$$

where $C_{\alpha\beta}$ is the Cotton tensor and μ is the so-called topological mass, see Refs. [15, 16, 17, 18] for definitions and conventions. The above relation is due to the fact that the only nonvanishing component of the Cotton tensor is given by $C_{uu} = y/(2l)\partial^3_{yyy}F$, and consequently the field equations involve the same pure radiation constraints (12). In what follows, we analyze different aspects of scalar configurations associated with the topologically massive gravity.

1. Vacuum AdS waves

Let us first consider the vacuum case $T_{\alpha\beta} = 0$ for which the only nontrivial equation is the uu one

$$\frac{1}{\mu}C_{uu} + G_{uu} - l^{-2}g_{uu} = \frac{y}{2l\mu}\partial_y\left(\frac{1}{y^{l\mu}}\partial_y\left(y^{l\mu}\partial_y F\right)\right) = 0. \tag{A.2}$$

The solution in this case, after an appropriate coordinate change in the same line of those used in the whole paper, reads

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\left(\frac{y}{l}\right)^{1-l\mu} du^2 - 2du dv + dy^2\right]. \tag{A.3}$$

The vacuum equation (A.2) allows two other solutions for the special values of the topological mass $\mu = \pm l^{-1}$, which are given in each case as

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\ln\left(\frac{y}{l}\right)du^2 - 2du dv + dy^2\right], \tag{A.4}$$

for $\mu = l^{-1}$, and

$$ds^2 = \frac{l^2}{y^2}\left[-F_1(u)\ln\left(\frac{y}{l}\right)y^2 du^2 - 2du dv + dy^2\right], \tag{A.5}$$

for $\mu = -l^{-1}$. The AdS waves (A.3) and (A.5) were derived previously in Ref. [16] using a correspondence established in that work between the conformal gravity with a self-interacting conformal scalar source on one side and the topologically massive gravity with a negative cosmological constant on the other side. Their corresponding configurations in conformal gravity are *pp* wave backgrounds. The solution (A.4) has no analog in the conformal gravity side, since it would correspond to a negative gravitational constant in this theory, see Ref. [16].

The structural function of the generic vacuum AdS waves (A.3) satisfies the Klein-Gordon equation

$$\Box F = {\mu_{\rm eff}}^2 F, \tag{A.6}$$

with an effective mass defined by $\mu_{\rm eff} = \mu\sqrt{1-(l\mu)^{-2}}$. This means that the effect of the negative cosmological constant is to lower the value of the physical topological mass. For the special values $\mu = \pm l^{-1}$ the corresponding structural functions in the AdS waves (A.4) and (A.5) do not satisfy a Klein-Gordon equation, which indicates that the theory does not have a massive character for these critical values.

2. AdS waves for a minimal scalar field

We study now the AdS waves of topologically massive gravity for which the source is a minimally coupled scalar field. For $\xi = 0$ the pure radiation constraints (12) imply that the scalar field only depends on the retarded time, $\Phi = \Phi(u)$, and additionally there is no self-interaction in this case, $U(\Phi) = 0$. The remaining Einstein equation is given by the $uu-$component

$$\frac{y}{2l\mu}\partial_y\left(\frac{1}{y^{l\mu}}\partial_y\left(y^{l\mu}\partial_y F\right)\right) = \kappa\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2, \tag{A.7}$$

whose solution, for a generic value of the topological mass $\mu \neq \pm l^{-1}$, reads

$$\begin{aligned} ds^2 &= \frac{l^2}{y^2}\left\{-\left[F_1(u)\left(\frac{y}{l}\right)^{1-l\mu} + \frac{\kappa l\mu}{1+l\mu}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2\ln\left(\frac{y}{l}\right)y^2\right]du^2 - 2du dv + dy^2\right\}, && \text{(A.8a)}\\ \Phi &= \Phi(u). && \text{(A.8b)}\end{aligned}$$

It easy to note that the above configuration is just a superposition of the vacuum AdS wave of topologically massive gravity (A.3) and the AdS wave of Einstein gravity obtained by considering a minimally coupled scalar field as a source (35) with an effective gravitational constant $\kappa_{\rm eff} = \kappa/[1+(l\mu)^{-1}]$. The structural function satisfies

$$\Box F = {\mu_{\rm eff}}^2 F + (\ldots), \tag{A.9}$$

where the terms within $(\ldots)$ depend nonlinearly on F. Here the effective topological mass is the same than in the vacuum case, i.e. $\mu_{\text{eff}} = \mu\sqrt{1-(l\mu)^{-2}}$.

For the critical values of the topological mass $\mu = l^{-1}$ and $\mu = -l^{-1}$, we have the following solutions

$$ds^2 = \frac{l^2}{y^2}\left\{-\left[F_1(u) + \frac{\kappa}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 y^2\right]\ln\left(\frac{y}{l}\right)du^2 - 2dudv + dy^2\right\}, \tag{A.10a}$$

$$\Phi = \Phi(u), \tag{A.10b}$$

and, respectively

$$ds^2 = \frac{l^2}{y^2}\left\{-\left[F_1(u) - \frac{\kappa l^2}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2 \ln\left(\frac{y}{l}\right)\right]\ln\left(\frac{y}{l}\right)\frac{y^2}{l^2}du^2 - 2dudv + dy^2\right\}, \tag{A.11a}$$

$$\Phi = \Phi(u). \tag{A.11b}$$

3. AdS waves for free nonminimally coupled scalar fields

We turn our attention now to the free scalar fields allowing nonminimal coupling ($\lambda = 0$ and $\xi \neq 0$) to an AdS wave. In this case the pure radiation constraints imply that the mass of this scalar field is fixed in terms of the nonminimal coupling by Eq. (22). In order to determine the AdS background allowed by this source it is useful to consider the following redefinitions

$$x = \frac{(1-4\xi)l\mu}{4}\left(\frac{y}{lf}\right)^{4\xi/(1-4\xi)}, \tag{A.12a}$$

$$H(u,x) = \frac{l^2 f^2}{y^2}F + l^2 f\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}, \tag{A.12b}$$

for which the $uu-$equation can be rewritten as

$$x^2\partial^3_{xxx}H - \frac{x[4\xi x-(1-4\xi)l\mu-3]}{4\xi}\partial^2_{xx}H - \frac{4\xi x-(1-2\xi)[(1-4\xi)l\mu+1]}{8\xi^2}\partial_x H - \frac{(1-4\xi)}{2\xi}H = 0. \tag{A.13}$$

We recognize the generalized hypergeometric differential equation whose general solution is expressed as

$$\begin{aligned} H(u,x) = {} & F_0(u)\left(\frac{4x}{(1-4\xi)l\mu}\right)^{-(1-4\xi)/(2\xi)} + F_1(u)\,{}_2\tilde{F}_2\left(1,\frac{1-4\xi}{2\xi};\frac{1-2\xi}{2\xi},\frac{1+(1-4\xi)l\mu}{4\xi};x\right) \\ & + F_3(u)\left(\frac{4x}{(1-4\xi)l\mu}\right)^{-(1-4\xi)(1+l\mu)/(4\xi)}\,{}_1\tilde{F}_1\left(\frac{(1-4\xi)(1-l\mu)}{4\xi};\frac{1-(1-4\xi)l\mu}{4\xi};x\right), \end{aligned} \tag{A.14}$$

where F_0, F_1, and F_3 are integration functions and ${}_1\tilde{F}_1(a;b;x)$ and ${}_2\tilde{F}_2(a,b;c,d;x)$ denote the corresponding generalized hypergeometric functions [23]. Returning to the original variables (A.12) and using the coordinate transformation (3) it is possible to fix the function f to

the unity and the function F_0 to zero. The resulting AdS wave configuration for a free nonminimally coupled scalar field in topological massive gravity is

$$
\begin{aligned}
ds^2 \;=\;& \frac{l^2}{y^2}\Bigg\{-\Bigg[F_1(u)\,{}_2\tilde{F}_2\left(1,\frac{1-4\xi}{2\xi};\frac{1-2\xi}{2\xi},\frac{1+(1-4\xi)l\mu}{4\xi};\frac{(1-4\xi)l\mu\kappa\Phi^2}{4}\right)\frac{y^2}{l^2}\\
&+F_3(u)\,{}_1\tilde{F}_1\left(\frac{(1-4\xi)(1-l\mu)}{4\xi};\frac{1-(1-4\xi)l\mu}{4\xi};\frac{(1-4\xi)l\mu\kappa\Phi^2}{4}\right)\left(\frac{y}{l}\right)^{1-l\mu}\Bigg]du^2\\
&-2dudv+dy^2\Bigg\}, \qquad \text{(A.15a)}\\
\Phi \;=\;& \frac{1}{\sqrt{\kappa}}\left(\frac{y}{l}\right)^{2\xi/(1-4\xi)}, \qquad \text{(A.15b)}
\end{aligned}
$$

where the functions F_1 and F_3 have been rescaled appropriately. Here the case $\xi = 1/4$ is excluded since, as in the Einstein gravity case, it does not allow free configurations.

4. Small topological mass limit: AdS waves for Conformal Gravity

Conformal gravity is a three-dimensional gravity theory just rigged by the Cotton tensor. In Refs. [15, 16, 17, 18] the *pp* waves of this theory with nonminimally coupled scalar sources have been studied. We analyze the AdS wave solutions of this theory using the fact that conformal gravity can be obtained from topologically massive gravity in the limit of small topological mass. More precisely, for $\mu \to 0$ with $\mu\kappa \sim 1$ the topologically massive gravity equations reduce to

$$C_{\alpha\beta} = \tilde{\kappa} T_{\alpha\beta}, \qquad \text{(A.16)}$$

where $\tilde{\kappa} = \mu\kappa$ is a dimensionless gravitational constant. The idea is to apply the above limit to the topologically massive gravity AdS waves obtained previously in order to obtain the corresponding AdS waves of conformal gravity.

In the minimal case $\xi = 0$, the AdS wave configurations (A.8) at the small topological mass limit yields to

$$
\begin{aligned}
ds^2 \;=\;& \frac{l^2}{y^2}\left\{-\left[F_1(u)+\tilde{\kappa}l^2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)^2\ln\left(\frac{y}{l}\right)y\right]\frac{y}{l}du^2-2dudv+dy^2\right\}, \qquad \text{(A.17a)}\\
\Phi \;=\;& \Phi(u). \qquad \text{(A.17b)}
\end{aligned}
$$

On the other hand, one can solve the conformal gravity (A.16) and conclude that the configuration (A.17) is the general solution for a minimally coupled scalar field.

A similar construction is applied in the case of a free nonminimally coupled scalar field ($\lambda = 0$). At the small topological mass limit one obtains from the solution (A.15),

$$
\begin{aligned}
ds^2 \;=\;& \frac{l^2}{y^2}\Bigg\{-\Bigg[F_1(u)\,{}_2\tilde{F}_2\left(1,\frac{1-4\xi}{2\xi};\frac{1-2\xi}{2\xi},\frac{1}{4\xi};\frac{(1-4\xi)l\tilde{\kappa}\Phi^2}{4}\right)\frac{y}{l}\\
&+F_3(u)\,{}_1\tilde{F}_1\left(\frac{1-4\xi}{4\xi};\frac{1}{4\xi};\frac{(1-4\xi)l\tilde{\kappa}\Phi^2}{4}\right)\Bigg]\frac{y}{l}du^2-2dudv+dy^2\Bigg\}, \qquad \text{(A.18a)}\\
\Phi \;=\;& \frac{1}{\sqrt{l\tilde{\kappa}}}\left(\frac{y}{l}\right)^{2\xi/(1-4\xi)}, \qquad \text{(A.18b)}
\end{aligned}
$$

which turns to be also the general solution of the conformal gravity equations for a free nonminimally coupled scalar field.

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